ECE 201: Analytical Tools for ECE

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Chapter 4

Laplace Transforms

Laplace transforms belong to a class of integral transforms (others are Fourier transform, Mellin transform, Hankel transform, Steiltjes transform) that convert a constant coefficient differential equation in the independent variable \( t \) to an algebraic equation in a conjugate variable \( s \). If \( t \) stands for the time coordinate, then \( s \) is known as the complex-frequency. If \( x \) stands for the space coordinate, then \( s \) will be the complex wavenumber and so on. The algebraic equation one ends up getting relates the transform of the dependent variable \( Y(s) \) to the transform of the source function and to the initial conditions. With the Laplace transform approach, the complete solution is obtained directly in one step without having to separately find the homogeneous solutions, the particular solution, and having to enforce the initial conditions at the end.

4.1 Forward and Inverse Laplace Transforms

**Definition 4.1.1 (Laplace Transform).** Given a function \( f(t), t \in (0, \infty) \), the Laplace transform \( F(s) \), which is a complex-valued function in the complex variable \( s = \sigma + i\omega \), is defined as

\[
F(s) = \mathcal{L}[f(t)] = \int_{t=0}^{\infty} f(t)e^{-st} \, dt.
\]  

(4.1)

Symbolically we write \( f(t) \Leftrightarrow F(s) \).

If \( |f(t)| \) behaves as \( e^{kt} \) for large \( t \), where \( k \) is a real number, then we require \( \sigma = \Re(s) > k \) for the integral (4.1) to converge (that is, yield a finite value). The region \( \sigma > k \) is known as the *region of existence* of the Laplace transform.

As a example if \( f(t) = Ae^{-at}, \ a > 0 \), then

\[
F(s) = \int_{0}^{\infty} Ae^{-at} e^{-st} \, dt = A \int_{0}^{\infty} e^{-(s+a)t} \, dt = \frac{-A}{(s+a)} e^{-(s+a)t} \bigg|_{0}^{\infty} = \frac{A}{(s+a)}, \ \sigma > -a.
\]

Here \( k = -a \) and the region of existence of the Laplace transform is \( \sigma > -a \).
Definition 4.1.2 (Inverse Laplace Transform). The inverse Laplace transform is defined as

\[ f(t) = \mathcal{L}^{-1} [F(s)] = \frac{1}{2\pi i} \int_{s=c-i^{\infty}}^{c+i^{\infty}} F(s)e^{st} \, ds, \quad c \text{ real} > k. \]  

(4.2)

![Figure 4.1: Contour for the inverse Laplace transform. Shaded area is \( \sigma = \Re(s) > k \).](image)

The inverse transform enables one to get \( f(t) \) given \( F(s) \). In practice we seldom use the contour integral in (4.2) to determine \( f(t) \) from \( F(s) \). Instead, we will use look-up table to relate an \( F(s) \) to the corresponding \( f(t) \).

4.2 Important Properties of Laplace Transform

We first discuss some of the most important properties of Laplace transform that are directly applicable to the solution of differential equations.

(i) **Linearity:** If a function \( f_1(t) \) has a Laplace transform \( F_1(s) \) and if a function \( f_2(t) \) has a Laplace transform \( F_2(s) \), then in the common region of existence, the Laplace transform of a linear combination is

\[ \mathcal{L} [c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s), \]

where \( c_1 \) and \( c_2 \) are complex constants.

As an example, (i) \( \mathcal{L} [e^{-t}] = \frac{1}{s+1}, \quad \Re(s) > -1 \), (ii) \( \mathcal{L} [e^{-3t}] = \frac{1}{s+3}, \quad \Re(s) > -3 \), then \( \mathcal{L} [e^{-t} + (1-i)e^{-3t}] = \frac{1}{s+1} + \frac{1-i}{s+3} \) in the common region of existence, \( \text{viz.}, \) for \( \Re(s) > -1 \).
(ii) **Derivative**: If $F(s)$ is the Laplace transform of a function $f(t)$, then the Laplace transform of its derivative is

$$
\mathcal{L} \left[ f'(t) \right] = \int_0^\infty f'(t)e^{-st} dt = f(t)e^{-st}\bigg|_0^\infty - (-s) \int_0^\infty f(t)e^{-st} dt = sF(s) - f(0). \quad (4.3)
$$

(iii) **Second Derivative**: Applying the property (4.3) to the second derivative we arrive at

$$
\mathcal{L} \left[ f''(t) \right] = \int_0^\infty f''(t)e^{-st} dt = s\mathcal{L} \left[ f'(t) \right] - f'(0) = s^2F(s) - sf(0) - f'(0). \quad (4.4)
$$

(iv) **Integral**: If $F(s)$ is the Laplace transform of a function $f(t)$, then the Laplace transform of the integral of $f(t)$ is

$$
\mathcal{L} \left[ \int_0^t f(\tau) d\tau \right] = \int_0^\infty \left[ \int_0^t f(\tau) d\tau \right] e^{-st} dt
$$

$$
= \left[ \int_0^t f(\tau) d\tau \right] \left. \frac{e^{-st}}{-s} \right|_{t=0}^{t=\infty} - \frac{1}{-s} \int_0^\infty f(t)e^{-st} dt
$$

$$
= \frac{F(s)}{s}, \quad (4.5)
$$

provided that $\int_0^\infty f(\tau) d\tau$ is finite.

Applying these properties to linear, constant coefficient ODEs we get

$$
\mathcal{L} \left[ \frac{dy}{dt} + ay(t) = r(t) \right] \implies sY(s) - y(0) + aY(s) = R(s)
$$

$$
\implies (s+a)Y(s) = R(s) + y(0)
$$

$$
\implies Y(s) = \frac{R(s) + y(0)}{s+a}. \quad (4.6)
$$

For the second order ODE $L[y''(t) + ay'(t) + by(t) = r(t)] \implies$

$$
\begin{align*}
& s^2Y(s) - sy(0) - y'(0) + a[sY(s) - y(0)] + bY(s) = R(s) \\
& \implies (s^2 + as + b)Y(s) = R(s) + (s + a)y(0) + y'(0) \\
& \implies Y(s) = \frac{R(s) + (s + a)y(0) + y'(0)}{s^2 + as + b}. \quad (4.7)
\end{align*}
$$

It is seen that the Laplace transform, $Y(s)$, of the unknown function $y(t)$ is related to the Laplace transform, $R(s)$, of the source function $r(t)$ algebraically. This is one of the main advantages of using Laplace transform. Furthermore, note that the initial conditions $y(0)$ and $y'(0)$ are included automatically in the approach and appear alongside $R(s)$ as additional source functions for $Y(s)$. 

Definition 4.2.1 (Pole Singularity). If a function $F(s)$ behaves as
\[
F(s) \sim \frac{1}{(s - s_p)^n}, \quad n \geq 1 \text{ an integer}
\]
in the vicinity of a complex number $s_p$, then $s = s_p$ is called a pole of order $n$ of the function $F(s)$.

4.3 Laplace Transform of Some Common Functions

4.3.0.1 Unit Step Function $\theta(t)$

Consider the unit step function
\[
\theta(t) = \begin{cases} 
1, & t > 0 \\
0, & t < 0 
\end{cases},
\]
shown in Fig. 4.2(a). Its Laplace transform is

\[
\Theta(s) = \mathcal{L}\left[\theta(t)\right] = \int_0^\infty e^{-st} dt = \frac{e^{-st}}{-s}\bigg|_0^\infty = \frac{1}{s}, \quad \Re(s) > 0.
\]

The transform has a pole of order 1 at the origin.

Consider now a delayed step function $\theta(t - t_0)$, $t_0 > 0$, Fig. 4.2(b). Its Laplace transform is

\[
\mathcal{L}\left[\theta(t - t_0)\right] = \int_{t_0}^\infty e^{-st} ds = \frac{e^{-st}}{-s}\bigg|_{t_0}^\infty = \frac{e^{-st_0}}{s}, \quad \Re(s) > 0.
\]

It is seen that $\mathcal{L}\left[\theta(t - t_0)\right] = e^{-st_0}\Theta(s)$, a property which will continue to hold for all delayed functions. Indeed, since $f(t)$ is assumed zero for $t < 0$, we have

\[
\mathcal{L}\left[f(t - t_0)\right] = \int_{t=t_0}^\infty f(t - t_0)e^{-st} dt = e^{-st_0} \int_{t-t_0=0}^\infty f(t - t_0)e^{-s(t-t_0)} dt
\]

\[
= e^{-st_0}\mathcal{L}\left[f(t)\right] = e^{-st_0}F(s).
\]
Example 4.3.0.1: Find the Laplace transform of the pulse function \( d(t) = Ar_w(t) := A\theta(t) - A\theta(t-w), A \) a constant, Fig. 4.3.

![Rectangular pulse function](image)

Figure 4.3: Rectangular pulse function of height \( A \) and width \( w \).

The pulse function is constructed as the difference between a step function and a delayed version of it, both having the same amplitude. We use the linearity property and the above results for a delayed function to get

\[
\mathcal{L}[d(t)] = \frac{A}{s} - \frac{Ae^{-sw}}{s} = \frac{A(1 - e^{-sw})}{s} \quad (4.10)
\]

\[
\sim Aw = \text{Area as } w \to 0. \quad (4.11)
\]

4.3.0.2 Delta Function \( \delta(t) \)

Definition 4.3.1 (Delta function a.k.a. Impulse function). A delta function \( \delta(t) \) is defined as the limiting case of the pulse function \( d(t) = Ar_w(t) \) as \( w \to 0 \) for \( A = 1/w \).

In this case the area \( Aw \) under the pulse function becomes unity and by (4.9), the Laplace transform of the delta function is

\[
\mathcal{L}[\delta(t)] = 1 \quad (4.12)
\]

What this result tells us is that the impulse function which only lasts momentarily in the time-domain contains all complex frequencies with the same amplitude. An approximate version of delta function is experienced in real life in the form of lightning, which is known to affect radio reception at all frequencies.

4.3.0.3 Power Function \( p_n(t) \)

For the power function of degree \( n, n \) a non-negative integer (see Fig. 4.4), defined as

\[
p_n(t) = \begin{cases} 
  t^n, & t > 0 \\
  0, & t < 0 
\end{cases} 
\quad (4.13)
\]

the Laplace transform using integration by parts is

\[
P_n(s) = \int_0^\infty p_n(t)e^{-st} \, dt = \left. t^n e^{-st} \right|_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt = \frac{n}{s} P_{n-1}(s).
\]
We therefore have a recursive relation for the Laplace transform of a power function of degree \( n \) in terms of the Laplace transform of a power function of degree \( n - 1 \). Note that \( p_0(t) = \theta(t) \) so that \( P_0(s) = 1/s \). Therefore, \( P_1(s) = \frac{1}{s} P_0(s) = 1/s^2 \), \( P_2(s) = \frac{2}{s} P_1(s) = 2!/s^3 \), \( P_3 = \frac{3}{s} P_2(s) = 3!/s^4 \), \ldots, \( P_n = n!/s^{n+1} \). To summarize, the Laplace transform of a power function of degree \( n \) is

\[
P_n(s) = \frac{n!}{s^{n+1}}, \quad n \geq 0, \quad \Re(s) > 0 \tag{4.14}
\]

and is seen to have a pole of order \( n + 1 \) at the origin. Note that higher the variation of the function in the time-domain, the higher the order of the pole at the origin is in the complex frequency domain.

### 4.3.0.4 Complex exponentials \( e^{i\omega_0 t} \)

For a one-sided complex exponential \( \theta(t)e^{i\omega_0 t} \), \( \omega_0 \) real, the Laplace transform is

\[
\mathcal{L} \left[ \theta(t)e^{i\omega_0 t} \right] = \int_0^\infty e^{-(s-i\omega_0)t} \, dt = \left. \frac{e^{-(s-i\omega_0)t}}{-(s-i\omega_0)} \right|_0^\infty = \frac{1}{s - i\omega_0}, \quad \Re(s) > 0. \tag{4.15}
\]

This result is true irrespective of whether \( \omega_0 \) is positive or negative. Since \( \cos\omega_0 t = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \) and \( \sin\omega_0 t = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \), we have using the property of linearity that

\[
\mathcal{L} \left[ \theta(t) \cos\omega_0 t \right] = \frac{1}{2} \left[ \frac{1}{s - i\omega_0} + \frac{1}{s + i\omega_0} \right] = \frac{s}{s^2 + \omega_0^2}, \quad \Re(s) > 0, \tag{4.16}
\]

\[
\mathcal{L} \left[ \theta(t) \sin\omega_0 t \right] = \frac{1}{2i} \left[ \frac{1}{s - i\omega_0} - \frac{1}{s + i\omega_0} \right] = \frac{\omega_0}{s^2 + \omega_0^2}, \quad \Re(s) > 0. \tag{4.17}
\]

**Example 4.3.0.2:** Determine the Laplace transform of the functions \( f_c(t) = \theta(t)e^{-at} \cos\omega_0 t \) and \( f_s(t) = \theta(t)e^{-at} \sin\omega_0 t, \ a > 0. \)
We first find the Laplace transform of \(f_e(t;\omega_0) = \theta(t)e^{-at}e^{i\omega_0 t}\).

\[
\mathcal{L}[f_e(t;\omega_0)] = \mathcal{L}[\theta(t)e^{-at}e^{i\omega_0 t}] = \int_0^\infty e^{-at}e^{i\omega_0 t}e^{-st}dt
\]

\[
= \left. \frac{e^{-(s+a-i\omega_0)t}}{-s+a-i\omega_0} \right|_0^\infty = \frac{1}{s+a-i\omega_0}, \Re(s) > -a. \quad (4.18)
\]

We use this to arrive at the following results

\[
\mathcal{L}[f_c(t;\omega_0)] = \mathcal{L}[\theta(t)e^{-at}\cos\omega_0 t] = \frac{1}{2} \left[ \frac{1}{s+a-i\omega_0} + \frac{1}{s+a+i\omega_0} \right], \Re(s) > -a; \quad (4.19)
\]

\[
\mathcal{L}[f_s(t;\omega_0)] = \mathcal{L}[\theta(t)e^{-at}\sin\omega_0 t] = \frac{1}{2i} \left[ \frac{1}{(s+a) - i\omega_0} - \frac{1}{(s+a) + i\omega_0} \right], \Re(s) > -a; \quad (4.20)
\]

\[
\mathcal{L}[f_c(t-t_0;\omega_0)] = \frac{(s+a)e^{-st_0}}{(s+a)^2 + \omega_0^2}, \Re(s) > -a; \quad (4.21)
\]

\[
\mathcal{L}[f_s(t-t_0;\omega_0)] = \frac{\omega_0 e^{-st_0}}{(s+a)^2 + \omega_0^2}, \Re(s) > -a. \quad (4.22)
\]

Summarizing from all these examples leads us to the entries of Table 4.1.

### 4.4 Application of Laplace Transform to Constant Coefficient ODEs

#### 4.4.1 First-Order Equations

Consider the ODE \(y'(t) + ay(t) = r(t), \ t > 0, \ y(t = 0) = y_0\). Taking the Laplace transform of this equation results in the algebraic equation (4.6) for the transform \(Y(s)\):

\[
Y(s) = \frac{R(s) + y_0}{s+a}. \quad (4.23)
\]

The unknown function in the time-domain is

\[
y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[ \frac{R(s) + y_0}{s+a} \right]. \quad (4.24)
\]

**Example 4.4.1.1:** Consider the first-order \(RL\) circuit shown in Fig. 4.5 fed by means of a voltage source \(v(t)\). The circuit elements are \(R = 1\ \Omega, \ L = 0.5\ H\). The differential equation
Table 4.1: Laplace Transform Pairs and Properties

<table>
<thead>
<tr>
<th>Case</th>
<th>Function $f(t)$, $t &gt; 0$</th>
<th>$F(s) = \mathcal{L} \left[ f(t) \right]$, $\Re(s) &gt; k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e^{-at}$</td>
<td>$\frac{1}{s+a}$, $\Re(s) &gt; -a$</td>
</tr>
<tr>
<td>2</td>
<td>$\theta(t)$</td>
<td>$\frac{1}{s}$, $\Re(s) &gt; 0$</td>
</tr>
<tr>
<td>3</td>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$, $\Re(s) &gt; 0$</td>
</tr>
<tr>
<td>5</td>
<td>$\cos \omega_0 t$</td>
<td>$\frac{s}{s^2 + \omega_0^2}$, $\Re(s) &gt; 0$</td>
</tr>
<tr>
<td>6</td>
<td>$\sin \omega_0 t$</td>
<td>$\frac{\omega_0}{s^2 + \omega_0^2}$, $\Re(s) &gt; 0$</td>
</tr>
<tr>
<td>7</td>
<td>$e^{-at} \cos \omega_0 t$</td>
<td>$\frac{(s+a)}{(s+a)^2 + \omega_0^2}$, $\Re(s) &gt; -a$</td>
</tr>
<tr>
<td>8</td>
<td>$e^{-at} \sin \omega_0 t$</td>
<td>$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$, $\Re(s) &gt; -a$</td>
</tr>
<tr>
<td>9</td>
<td>$f(t-t_0)$, $t_0 &gt; 0$</td>
<td>$e^{-st_0}F(s)$, $\Re(s) &gt; k$</td>
</tr>
<tr>
<td>10</td>
<td>$f'(t)$</td>
<td>$sF(s) - f(0)$</td>
</tr>
<tr>
<td>11</td>
<td>$e^{-at}f(t)$</td>
<td>$F(s + a)$</td>
</tr>
</tbody>
</table>
for the circuit current is given by

\[ L \frac{di_L}{dt} + Ri_L = v(t) \implies \frac{di_L}{dt} + 2i_L = 2v(t) \]

with the initial condition \( i_L(t = 0) = i_0 = 1 \text{ A} \). We will find the current in the three separate cases (i) \( v(t) = \delta(t) \), (ii) \( v(t) = \theta(t) \), and (iii) \( v(t) = \cos \omega_0 t \), \( \omega_0 = 2\pi \).

![RL-circuit fed by an ac source \( v(t) \)](image)

Let \( \mathcal{L}[i_L(t)] = I_L(s) \), \( \mathcal{L}[v(t)] = V(s) \). From Table 4.1, the Laplace transform of the voltage source in the three cases is (i) \( V(s) = 1 \), (ii) \( V(s) = 1/s \), (iii) \( V(s) = s/(s^2 + 4\pi^2) \).

From (4.6) the Laplace transform of the current in the three cases is

\[
I_L(s) = \frac{2V(s) + i_0}{s + 2} = \begin{cases} 
\frac{2 + 1}{s + 2} = \frac{3}{s + 2}, & \text{case (i)} \\
\frac{2s + 1}{s + 2} = \frac{1}{s}, & \text{case (ii)} \\
\frac{2s}{s^2 + 4\pi^2} + 1 = \left[ \frac{s}{s^2 + 4\pi^2} + \frac{2\pi^2}{s^2 + 4\pi^2} + \frac{\pi^2}{s + 2} \right] \frac{1}{1 + \pi^2}, & \text{case (iii)}
\end{cases}
\]

The time-domain current is obtained by taking the inverse Laplace transform of the above results. The individual terms in the three separate cases can be identified with entries in Table 4.1 to result in

\[
i_L(t) = \begin{cases} 
3e^{-2t} \theta(t), & \text{case (i)} \\
\theta(t), & \text{case (ii)} \\
\left[ \cos 2\pi t + \pi \sin 2\pi t + \pi^2 e^{-2t} \right] \theta(t), & \text{case (iii)}
\end{cases}
\]

(4.25)

Note that the above solution for the current momentarily after \( t = 0 \) gives \( i_L(t = 0^+) = 3 \) for case (i), while it gives \( i_L(t = 0^+) = 1 \) for cases (ii) and (iii). The reason for the departure from the given initial current momentarily before \( t = 0 \), namely \( i_L(0^-) = 1 \) in case (i) is due to the fact that, in general, the differential equation suggests that \( i_L(t = 0^+) = i_L(t = 0^-) + 2 \int_{t=0^-}^{0^+} v(t) \, dt \). In the case of impulsive source, the integral of voltage is equal to unity giving \( i_L(0^+) = 1 + 2 = 3 \). In the case of non-impulsive sources as in cases (ii) and (iii), the integral of the voltage equals zero giving \( i_L(0^+) = i_L(0^-) = 1 \).
4.4. APPLICATION OF LAPLACE TRANSFORM TO CONSTANT COEFFICIENT ODES

4.4.1.1 A Note on Partial Fraction Expansions

We consider here the expansion of proper rational functions (degree of numerator polynomial less than the degree of denominator polynomial) by means of partial fractions. Let \( G(s) = \frac{(s^2 + as + b)}{(s^2 + \omega_0^2)(s + \alpha)} \) be given. We would like to express it in a partial fraction of the form \( \frac{(As + B)}{(s^2 + \omega_0^2)} + \frac{C}{(s + \alpha)} \) and determine the constants \( A, B, \) and \( C \) in a consistent manner. Accordingly, we let

\[
G(s) = \frac{(s^2 + as + b)}{(s^2 + \omega_0^2)(s + \alpha)} = \frac{(As + B)}{(s^2 + \omega_0^2)} + \frac{C}{(s + \alpha)} \tag{4.26}
\]

\[
= \frac{(A + C)s^2 + (\alpha A + B)s + \alpha B + \omega_0^2 C}{(s^2 + \omega_0^2)(s + \alpha)} \tag{4.27}
\]

Method-1:

Comparing the coefficients of \( s^2, s^1 \) and \( s^0 \) in the numerators of (4.26) and (4.27) gives the three equations in the three unknowns \( A, B, \) and \( C \):

\[
\begin{align*}
A + C &= 1, \\
\alpha A + B &= a, \\
\alpha B + \omega_0^2 C &= b
\end{align*}
\]

\[
\begin{bmatrix} 1 & 0 & 1 \\ \alpha & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \tag{4.28}
\]

The matrix equation can be solved either by Cramer’s rule or matrix inverse or Gaussian elimination.

Method-2:

Multiplying both sides of equation (4.26) by \( (s + \alpha) \) and evaluating at \( s = -\alpha \) gives

\[
C = \lim_{s \to -\alpha} [(s + \alpha)G(s)] = \frac{(\alpha^2 - a\alpha + b)}{\alpha^2 + \omega_0^2} \tag{4.29}
\]

Recalling that \( s^2 + \omega_0^2 = (s - i\omega_0)(s + i\omega_0) \) both sides of (4.26) by \( (s \pm i\omega_0) \) and evaluating at \( s = \mp i\omega_0 \) gives

\[
\begin{align*}
-i\omega_0A + B &= -\frac{-\omega_0^2 - ai\omega_0 + b}{(-i\omega_0 + \alpha)} \\
i\omega_0A + B &= -\frac{-\omega_0^2 + ai\omega_0 + b}{(i\omega_0 + \alpha)}
\end{align*}
\]

Adding and subtracting these two equation gives

\[
\begin{align*}
B &= \frac{1}{2} \left[ -\frac{-\omega_0^2 - ai\omega_0 + b}{(-i\omega_0 + \alpha)} + \frac{-\omega_0^2 + ai\omega_0 + b}{(i\omega_0 + \alpha)} \right] = \frac{\omega_0^2(a - \alpha) + \alpha b}{\omega_0^2 + \alpha^2}, \tag{4.30}
\end{align*}
\]

\[
A = \frac{1}{2i\omega_0} \left[ -\frac{-\omega_0^2 + ai\omega_0 + b}{(i\omega_0 + \alpha)} + \frac{-\omega_0^2 - ai\omega_0 - b}{(-i\omega_0 + \alpha)} \right] = \frac{\omega_0^2 + a\alpha - b}{\omega_0^2 + \alpha^2}. \tag{4.31}
\]

\[\]
Example 4.4.1.2: Find the partial fraction expansion of \( G(s) = \frac{s^2 + 2s + 4\pi^2}{(s^2 + 4\pi^2)(s + 2)} \), which was encountered in Example 4.4.1.1. Determine its Laplace inverse.

Here \( a = 2, b = 4\pi^2, \omega_0 = 2\pi, \alpha = 2 \). The linear system (4.28) reduces to

\[
\begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & 0 \\
0 & 2 & 4\pi^2
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
4\pi^2
\end{bmatrix}.
\]

Solving the linear system gives \( A = 1/(\pi^2 + 1), B = 2\pi^2/(\pi^2 + 1), \) and \( C = \pi^2/(\pi^2 + 1) \) so that

\[
G(s) = \frac{s^2 + 2s + 4\pi^2}{(s^2 + 4\pi^2)(s + 2)} = \frac{1}{1 + \pi^2} \left( \frac{s}{s^2 + 4\pi^2} + \frac{2\pi^2}{s^2 + 4\pi^2 + 2\pi^2} + \frac{\pi^2}{s + 2} \right).
\]

Note that equations (4.29), (4.30), and (4.31) may also be used to determine \( C, B, \) and \( A \). Indeed they give \( C = (4 - 4 + 4\pi^2)/(4\pi^2 + 4) = \pi^2/(\pi^2 + 1), B = (4\pi^2 \cdot 0 + 8\pi^2)/(4\pi^2 + 4) = 2\pi^2/(\pi^2 + 1), A = (4\pi^2 + 4 - 4\pi^2)/(4\pi^2 + 4) = 1/(\pi^2 + 1) \), which are the same as those calculated by matrix inversion.

The inverse Laplace transform of \( G(s) \) using entries from the look-up Table 4.1 is

\[
g(t) = \frac{\cos 2\pi t}{1 + \pi^2} + \frac{\pi \sin 2\pi t}{1 + \pi^2} + \frac{\pi^2 e^{-2t}}{1 + \pi^2}.
\]

Example 4.4.1.3: Find the partial fraction expansion of \( G(s) = \frac{3s - 137}{(s^2 + 2s + 401)} \) and determine its Laplace inverse. The first step is to cast the denominator as a sum of squares such as those listed in items (7) and (8) in Table 4.1. Note that \( s^2 + 2s + 4001 = (s + 1)^2 + 20^2 \). Therefore,

\[
G(s) = \frac{3s - 137}{(s + 1)^2 + 20^2} = \frac{3(s + 1) - 140}{(s + 1)^2 + 20^2} = 3 \frac{(s + 1)}{(s + 1)^2 + 20^2} - 7 \frac{20}{(s + 1)^2 + 20^2},
\]

so that

\[
g(t) = \mathcal{L}^{-1} [G(s)] = 3e^{-t} \cos 20t - 7e^{-t} \sin 20t.
\]

4.4.2 Second Order Equations

It is seen from (4.7) that for the second order ODE \( y''(t) + ay'(t) + by(t) = r(t) \) with initial conditions \( y(0) = y_0, y'(0) \), the solution is

\[
y(t) = \mathcal{L}^{-1} \left[ \frac{R(s) + (s + a)y_0 + y'(0)}{s^2 + as + b} \right],
\]

where \( r(t) \leftrightarrow R(s) \).
Example 4.4.2.1: Consider a series RLC circuit fed by means of a voltage source \( v(t) \), Fig. 4.6. The differential equation for the capacitor voltage, \( v_c(t) \), is \( LCv''_c + RCv'_c + v_c = v(t) \). Let \( R = 200 \Omega, L = 10 \mu H, C = 1 \text{ nF}, v_c(0) = 0, i_L(0) = CV'_c(0) = 1 \text{ mA}. \) Let \( \omega_0 = 1/\sqrt{LC} = 10^7 \text{ rad/s} \) and \( \alpha = R/L = 2 \times 10^7 = 2\omega_0 \).

For the case of \( v(t) \equiv 0 \), the solution has already been found in Example 3.5.1.1 to be \( v_c(t) = 0.1 \omega_0 t e^{-\omega_0 t} \). Let us rework this specific case using the Laplace transform technique. With \( v(t) \equiv 0 \), the ODE becomes, after dividing throughout by \( LC \), \( v''_c + \alpha v'_c + \omega_0^2 v_c = 0 \). The initial conditions are \( v_c(0) = 0, v'_c(0) = 10^6 \text{ V-s}^{-1} \). Using (4.7) with \( y(t) \leftarrow v_c(t) \leftrightarrow V_c(s) \), \( y(0) \leftarrow v_c(0), y'(0) \leftarrow v'_c(0) \) gives

\[
V_c(s) = \frac{(s + \alpha)v_c(0) + v'_c(0)}{(s^2 + \alpha s + \omega_0^2)} = \frac{10^6}{(s + \omega_0)^2}, \text{ since } \alpha = 2\omega_0.
\]

We don’t have an entry for the Laplace inverse of \( 1/(s + \omega_0)^2 \) in Table 4.1. However, note from entries 11 and 4 that \( \mathcal{L}^{-1} \left[ 1/(s + \omega_0)^2 \right] = e^{-\omega_0 t} \mathcal{L}^{-1} \left[ 1/s^2 \right] = e^{-\omega_0 t} t \). Therefore

\[
v_c(t) = 10^6 t e^{-\omega_0 t} = 0.1 \omega_0 t e^{-\omega_0 t}, \quad (4.32)
\]

which is the same as that obtained in Example 3.5.1.1.

Let us now solve the problem with the same initial conditions, but for a non-zero, sinusoidal input of \( v(t) = V_0 \sin \omega_p t \), where \( \omega_p = 5 \times 10^6 = \omega_0/2 \text{ rad/s} \). From Table 4.1 \( V(s) = V_0\omega_p/(s^2 + \omega_p^2) \) and using \( R(s) \leftarrow V(s)/LC = \omega_0^2 V(s) \) in equation (4.7) gives

\[
V_c(s) = \frac{\omega_0^2 V(s) + v'_c(0)}{(s + \omega_0)^2} = \frac{V_0\omega_p^2 \omega_p}{(s + \omega_0)^2(s^2 + \omega_p^2)} + \frac{v'_c(0)}{(s + \omega_0)^2}.
\]

Let us focus on the first term as the Laplace inverse of the second term has already been determined above. We write it as a sum of two proper rational functions with unknown constants \( A, B, C, \) and \( D \)

\[
\frac{\omega_0^2 \omega_p}{(s^2 + \omega_p^2)(s + \omega_0)^2} = \frac{As + B}{(s^2 + \omega_p^2)(s + \omega_0)^2} + \frac{Cs + D}{(s + \omega_0)^2}
\]

\[
= \frac{(As + B)(s + \omega_0)^2 + (Cs + D)(s^2 + \omega_p^2)}{(s^2 + \omega_p^2)(s + \omega_0)^2}
\]

\[
= \frac{(A + C)s^3 + (2\omega_0 A + B + D)s^2 + (A\omega_0^2 + 2\omega_0 B + C\omega_p^2)s + (B\omega_0^2 + D\omega_p^2)}{(s^2 + \omega_p^2)(s + \omega_0)^2}
\]
Equating the numerators on both sides results in
\[ A + C = 0 \implies C = -A \]
\[ A\omega_0^2 + 2\omega_0B + C\omega_p^2 = 0 \implies 2\omega_0B = (\omega_0^2 - \omega_p^2)A \]
\[ 2\omega_0A + B + D = 0 \implies 2\omega_0D = -2\omega_0B - 4\omega_0^2A = -(3\omega_0^2 + \omega_p^2)A \]
\[ B\omega_0^2 + D\omega_p^2 = \omega_0^2\omega_p \implies A = -\frac{2\omega_0^3\omega_p}{(\omega_0^2 + \omega_p^2)^2} = -C \]

Therefore \[ B = \omega_0^2\omega_p(\omega_0^2 - \omega_p^2)/(\omega_0^2 + \omega_p^2)^2, \]
\[ D = \omega_0^2\omega_p(3\omega_0^2 + \omega_p^2)/(\omega_0^2 + \omega_p^2)^2. \]

Substituting the numerical values for \( \omega_0 = 10^7 \), \( \omega_p = 5 \times 10^6 \), the constants work out to be
\[ A = -0.64; \ B = 2.4 \times 10^6 = 0.48\omega_p; \ C = 0.64; \ D = 1.04 \times 10^7 = 1.04\omega_0. \]

The Laplace inverse is therefore
\[
\mathcal{L}^{-1}\left[ \frac{As + B}{(s^2 + \omega_p^2)} + \frac{Cs + D}{(s + \omega_0)^2} \right] = A \cos \omega_p t + \frac{B}{\omega_p} \sin \omega_p t + \\
\mathcal{L}^{-1}\left[ \frac{C(s + \omega_0) + D - C\omega_0}{(s + \omega_0)^2} \right] = -0.64 \cos \omega_p t + 0.48 \sin \omega_p t + 0.4 \omega_0 e^{-\omega_0 t}.
\]

Therefore combining this with (4.32) the capacitor voltage in the presence of the sinusoidal input is
\[ v_c(t) = \left[ -0.64 \cos \omega_p t + 0.48 \sin \omega_p t + 0.64 e^{-\omega_0 t} + 0.5 \omega_0 t e^{-\omega_0 t} \right] \theta(t) \]
and is plotted in Figure 4.7.

### 4.5 System of First Order Equations by Laplace Transform

We now touch upon a capstone topic that encompasses the four main pillars of the course: complex algebra, linear algebra, differential equations, and Laplace transform. We will only consider a 2 \times 2 system although the concepts carry over to arbitrary sizes. Consider the system of constant coefficient first order equations
\[
y'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = Ay(t) + g(t),
\]
where the matrix \( A \) is constant, together with the initial conditions \( y_1(0) = y_{10}, \ y_2(0) = y_{20}. \)

The quantities \( y_1(t) \) and \( y_2(t) \) may be thought of as the position and speed of an object
4.5. SYSTEM OF FIRST ORDER EQUATIONS BY LAPLACE TRANSFORM

moving in 1D, or the mesh currents in a cascaded first-order circuits, or the voltage and current in a transmission line. Our goal is to determine \( y(t) \) for \( t > 0 \). Let

\[
\mathcal{L}[y(t)] = \begin{bmatrix}
\mathcal{L}[y_1(t)] \\
\mathcal{L}[y_2(t)]
\end{bmatrix} = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \mathbf{Y}(s); \quad \mathcal{L}[g(t)] = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \mathbf{G}(s).
\]

From the derivative property of Laplace transform we have

\[
\mathcal{L}[y'(t)] = s\mathbf{Y}(s) - y_0 = s\mathbf{IY}(s) - y_0,
\]

where \( y_0 = [y_{01}, y_{02}]^T \), where \( \mathbf{I} \) is the \( 2 \times 2 \) identity matrix acting on the vector \( \mathbf{Y}(s) \). Taking the Laplace transform of (4.33) results in

\[
s\mathbf{IY}(s) - y_0 = \mathbf{AY}(s) + \mathbf{G}(s) \implies (s\mathbf{I} - \mathbf{A})\mathbf{Y}(s) = \mathbf{G}(s) + y_0.
\]

Writing \( \mathbf{B} = (s\mathbf{I} - \mathbf{A}) \) and inverting the matrix gives

\[
\mathbf{Y}(s) = \mathbf{B}^{-1}(\mathbf{G}(s) + y_0).
\]

Taking the inverse Laplace transform gives the solution

\[
y(t) = \mathcal{L}^{-1}\left[ \mathbf{B}^{-1}\left(\mathbf{G}(s) + y_0\right) \right]. \tag{4.34}
\]

**Example 4.5.0.1:** Consider two first-order circuits (an \( RL \) circuit and an \( RC \) circuit) cascaded to form a two-mesh network as shown in Fig. 4.8. The mesh currents are \( y_1(t) \) and \( y_2(t) \). Let the circuit be fed by a step function \( 12\theta(t) \). The initial mesh currents are given as \( y_1(0) = 0, y_2(0) = 0 \). Let the circuit elements be \( L = 1 \text{H}, C = \frac{1}{4} \text{F}, R_1 = 4 \Omega, R_2 = 6 \Omega \).

The differential equations for the two mesh currents can be obtained from Kirchhoff’s laws and are \( y_1' = -4y_1 + 4y_2 + 12, y_2' = -1.6y_1 + 1.2y_2 + 4.8 \). In a matrix form the equations can be written as

\[
y'(t) = \begin{bmatrix} -4 & 4 \\ -1.6 & 1.2 \end{bmatrix} y + \begin{bmatrix} 12\theta(t) \\ 4.8\theta(t) \end{bmatrix} = \mathbf{Ay}(t) + \mathbf{g}(t).
\]
Here
\[ G(s) = \begin{bmatrix} \frac{12}{s} \\ \frac{4.8}{s} \end{bmatrix} \quad B = sI - A = \begin{bmatrix} (s + 4) & -4 \\ 1.6 & (s - 1.2) \end{bmatrix}. \]

Note that \( \det B = (s + 4)(s - 1.2) + 6.4 = (s + 2)(s + 0.8) \) and so
\[
B^{-1} = \begin{bmatrix} \frac{(s - 1.2)}{(s + 2)(s + 0.8)} & \frac{4}{(s + 2)(s + 0.8)} \\ -1.6 & \frac{(s + 4)}{(s + 2)(s + 0.8)} \end{bmatrix}.
\]

The zeros of the determinant of \( B \) in the complex \( s \)-plane gives the eigenvalues of the matrix \( A \). Now
\[
B^{-1}(G(s) + y_0) = \begin{bmatrix} \frac{(s - 1.2)}{(s + 2)(s + 0.8)} & \frac{4}{(s + 2)(s + 0.8)} \\ -1.6 & \frac{(s + 4)}{(s + 2)(s + 0.8)} \end{bmatrix} \begin{bmatrix} \frac{12}{s} \\ \frac{4.8}{s} \end{bmatrix} = \begin{bmatrix} \frac{3}{s} - \frac{8}{(s + 2)} + \frac{5}{(s + 0.8)} \\ -\frac{4}{(s + 2)} + \frac{4}{(s + 0.8)} \end{bmatrix},
\]

where the last form is obtained by performing the partial fraction expansion. Taking the inverse Laplace transform we get the desired solution
\[
y(t) = \begin{bmatrix} 3 - 8e^{-2t} + 5e^{-0.8t} \\ -4e^{-2t} + 4e^{-0.8t} \end{bmatrix} \theta(t).
\]

Notice that \( y_1(0^+) = 0 = y_2(0^+) \). ■■